

# The residue theorem from a numerical perspective

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## Abstract

A short vignette illustrating Cauchy's integral theorem using numerical integration

*Keywords:* Residue theorem, Cauchy formula, Cauchy's integral formula, contour integration, complex integration, Cauchy's theorem.

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In this very short vignette, I will use contour integration to evaluate

$$\int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx \quad (1)$$

using numerical methods. This document is part of the **elliptic** package [Hankin \(2006\)](#).

The residue theorem tells us that the integral of  $f(z)$  along any closed nonintersecting path is equal to  $2\pi i$  times the sum of the residues inside it.

Take a semicircular path  $P$  from  $-R$  to  $+R$  along the real axis, then following a semicircle in the upper half plane, of radius  $R$  to close the loop. Now consider large  $R$ . Then  $P$  encloses a pole at  $i$  [there is one at  $-i$  also, but this is outside  $P$ , so irrelevant here] at which the residue is  $-i/2e$ . Thus

$$\oint_P f(z) dz = 2\pi i \cdot (-i/2e) = \pi/e \quad (2)$$

along  $P$ ; the contribution from the semicircle tends to zero as  $R \rightarrow \infty$ ; thus the integral along the real axis is the whole path integral, or  $\pi/e$ .

We can now reproduce this result analytically. First, choose  $R$ :

```
> R <- 400
```

And now define a path  $P$ . First, the semicircle:

```
> u1 <- function(x){R*exp(pi*1i*x)}  
> u1dash <- function(x){R*pi*1i*exp(pi*1i*x)}
```

and now the straight part along the real axis:

```
> u2 <- function(x){R*(2*x-1)}  
> u2dash <- function(x){R*2}
```

And define the function:

```
> f <- function(z){exp(1i*z)/(1+z^2)}
```

Now carry out the path integral. I'll do it explicitly, but note that the contribution from the first integral should be small:

```
> answer.approximate <-
+   integrate.contour(f,u1,u1dash) +
+   integrate.contour(f,u2,u2dash)
```

And compare with the analytical value:

```
> answer.exact <- pi/exp(1)
> abs(answer.approximate - answer.exact)
```

```
[1] 6.244969e-07
```

Now try the same thing but integrating over a triangle instead of a semicircle, using `integrate.segments()`. Use a path  $P'$  with base from  $-R$  to  $+R$  along the real axis, closed by two straight segments, one from  $+R$  to  $iR$ , the other from  $iR$  to  $-R$ :

```
> abs(integrate.segments(f,c(-R,R,1i*R))- answer.exact)
```

```
[1] 5.157772e-07
```

Observe how much better one can do by integrating over a big square instead:

```
> abs(integrate.segments(f,c(-R,R,R+1i*R, -R+1i*R))- answer.exact)
```

```
[1] 2.319341e-08
```

## 0.1. Residue theorem

Function `residue()` is a wrapper that takes a function  $f(z)$  and integrates  $f(z)/(z - z_0)$  around a closed loop which encloses  $z_0$ . If  $f(\cdot)$  is holomorphic within  $C$ , Cauchy's residue theorem states that

$$\oint_C \frac{f(z)}{z - z_0} = f(z_0) \quad (3)$$

and we can test this numerically:

```
> f <- function(z){sin(z)}
> numerical <- residue(f,z0=1,r=1)
> exact <- sin(1)
> abs(numerical-exact)
```

```
[1] 3.91766e-18
```

which is unreasonably accurate, IMO.

## References

Hankin RKS (2006). “Introducing elliptic, an R package for elliptic and modular functions.”  
*Journal of Statistical Software*, **15**(7).

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